Non-linearly elastic creasing

Dominic Emery
Supervisor: Prof. Yibin Fu
Keele UNIVERSITY

Introduction

Creasing as a bifurcation phenomenon has long been enigmatic. By definition, creases are a localisation of material *self contact* on the free surface of compressed elastic solids. The bifurcation condition for crease initiation can be sought by matching an "outer" uni-axially compressed state, an "inner" infinitesimal creased solution and an "intermediate" field of incremental displacement through a conservation law of energy and momentum.

The "outer" solution

We consider an incompressible, isotropic, hyper-elastic half space in *plane-strain* uni-axial compression. The reference and finitely deformed material configurations are denoted $\mathcal{B}_0$ and $\mathcal{B}_e$ respectively.

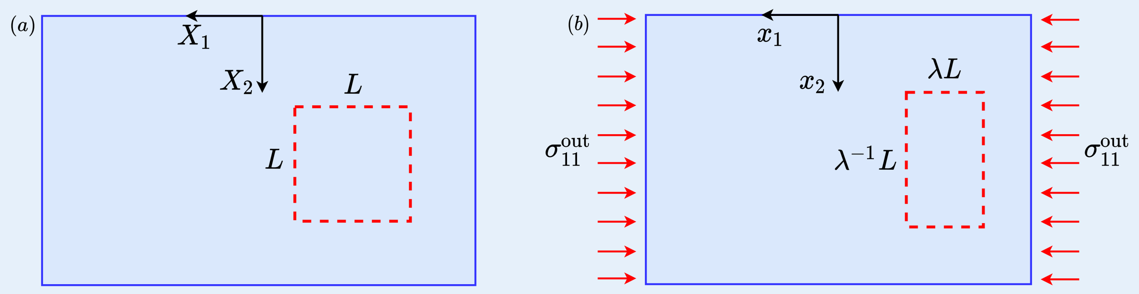

Figure 1: **(a)** The reference configuration $\mathcal{B}_0$. **(b)** The compressed configuration $\mathcal{B}_e$.

The mapping $\mathcal{B}_0 \to \mathcal{B}_e$ is enforced by a change of variables $x_1 = \lambda X_1$, $x_2 = \lambda^{-1} X_2$ and the *deformation gradient* is

$$\bar{\mathbf{F}} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \lambda\, \mathbf{e}_1 \otimes \mathbf{E}_1 + \lambda^{-1}\, \mathbf{e}_2 \otimes \mathbf{E}_2, \qquad (1)$$

where λ is the principle stretch. The Cauchy stress tensor is defined as $\sigma = 2\,\bar{W}'(I_1)\,\bar{\mathbf{F}}\bar{\mathbf{F}}^\top - p\mathbf{I}$, where p is the kinematic pressure, $\bar{W}$ is the strain-energy function and $I_1 = \mathrm{tr}\left(\bar{\mathbf{F}}\bar{\mathbf{F}}^\top\right)$. With use of (1) and the boundary condition $\sigma_{22}|_{x_2=0} = 0$, σ is found to have a single non-zero component

$$\sigma_{11}^{\text{out}} = 2\bar{W}'(I_1)\left(\lambda^2 - \lambda^{-2}\right). \qquad (2)$$

Thus, (2) gives the horizontal compressive stress enforcing the "outer" solution.

References

[1] Ciarletta, P., & Truskinovsky, L. (2019). Soft nucleation of an elastic crease. *Physical review letters*, 122(24), 248001.

[2] Noether, E. (1971). Invariant variation problems. *Transport Theory and Statistical Physics*, 1(3), 186-207.

The "inner" solution

At a critical compression λ_{cr}, an *infinitesimal* crease forms on the materials free surface. This "inner" deformation is a canonical mapping from the half space $\mathcal{B}_0^\star$ to the whole space $\mathcal{B}_e^\star$

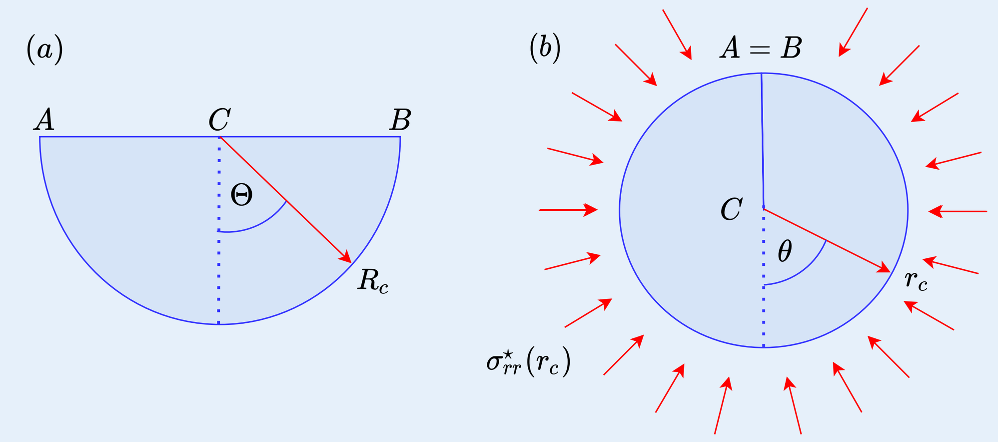

Figure 2: **(a)** The reference configuration $\mathcal{B}_0^\star$. **(b)** The finitely deformed configuration $\mathcal{B}_e^\star$.

enforced by the change in cylindrical co-ordinates $r = R/\sqrt{2}$, $\theta = 2\Theta$. The deformation gradient is then

$$\bar{\mathbf{F}}^\star = \frac{1}{\sqrt{2}}\mathbf{e}_r \otimes \mathbf{E}_r + \sqrt{2}\,\mathbf{e}_\theta \otimes \mathbf{E}_\Theta. \qquad (3)$$

Given self-contact length $r_c \ll 1$ and traction continuity across $\theta = \pm\pi$, we assume the equilibrium condition $\sigma_{11}^{\text{out}} = \sigma_{rr}(r_c)$. Thus, the pressure distribution in $\mathcal{B}_e^\star$ is:

$$p_{\text{in}}(r) = \bar{W}'\left(\frac{5}{2}\right) - 3\bar{W}'\left(\frac{5}{2}\right)\ln\left(\frac{r}{r_c}\right) - \sigma_{11}^{\text{out}}. \qquad (4)$$

Incremental displacement field

The crease formation imposes an incremental displacement field $\mathbf{u}$ onto $\mathcal{B}_e$ which decays to the uniform compression for $|\mathbf{x}| \gg 1$. Given its *localised* nature, the crease is analogous to a vertical point force whose magnitude $\delta f = -2r_c\,\sigma_{11}^{\text{out}}$ is the resultant along the boundary of the inner solution. By superposing this singular load onto $\mathcal{B}_e$, we find that

$$\mathbf{u} = (-\phi_{,2}, \phi_{,1}), \qquad \phi = \tilde{a}\,G\left(\tilde{x}_1, \tilde{x}_2\right) + \bar{a}\,G\left(\bar{x}_1, \bar{x}_2\right), \qquad (5)$$

is a scaled *biharmonic* solution to the incremental problem, where ϕ is a stream function introduced to satisfy the incompressibility constraint $u_{i,i} = 0$. The constants $\tilde{a}$ and $\bar{a}$ are deduced from traction-free boundary conditions and the Green's function $G(x, y) = x\ln\left(x^2 + y^2\right) + 2y\arctan\left(x/y\right)$ provides the desired singular behaviour and symmetry about $x_1 = 0$.

Bifurcation condition

The bifurcation condition for creasing gives the critical threshold for the coexistence of the 3 solutions, and is sought by minimising $\mathcal{E} = \int_{\mathcal{B}_0} \bar{W}(\mathbf{F})\,d\mathbf{X}$. Under spatial translation in $\mathbf{x}$, $\delta\mathcal{E} = 0$ due to homogeneity of the "outer" solution. By Noether's Theorem [2], this symmetry gives rise to the conservation law of energy and momentum:

$$\nabla \cdot \Sigma = 0 \implies \mathcal{J} \equiv \int_\eta \Sigma\mathbf{n}\,dS = \mathbf{0} \qquad (6)$$

where $\Sigma = \bar{W}\mathbf{I} - \mathbf{F}^{-1}\sigma\mathbf{F}$ is the Energy-Momentum tensor and $\mathbf{n}$ is the outward normal to the closed contour η.

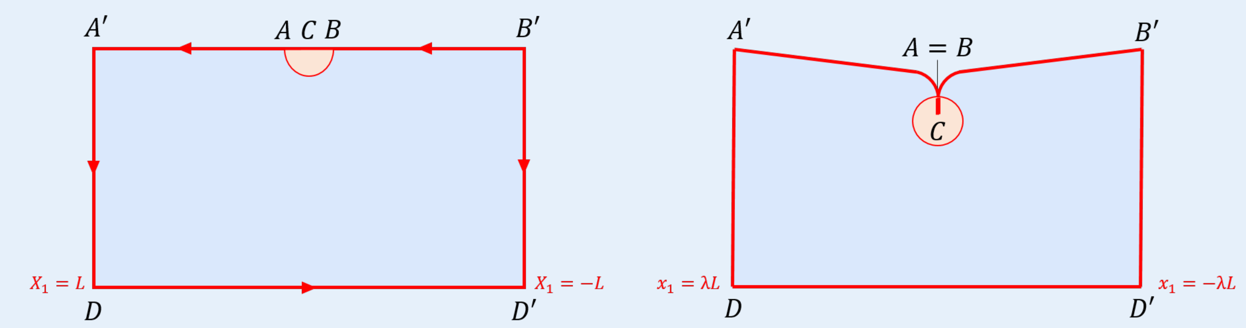

Figure 3: The contour η (red boundary) enclosing the "inner" and "incremental" regions. $\mathcal{J}$ must be globally zero along the entirety of η.

From (6), the following bifurcation condition for creasing in terms of a general strain-energy function is obtained

$$R_c\left(\bar{W}\left(5/2\right) - \sigma_{11}^{\text{out}}\right) + \lambda^{-1}\sigma_{11}^{\text{out}}\phi_{,2}|_{x_2=0} = 0.$$

Results

The Gent strain-energy function:

$$\bar{W}(I_1) = -\frac{\mu\,J_m}{2}\ln\left(1 - \frac{I_1 - 2}{J_m}\right), \qquad (7)$$

incorporates material strain stiffening via the constant J_m.

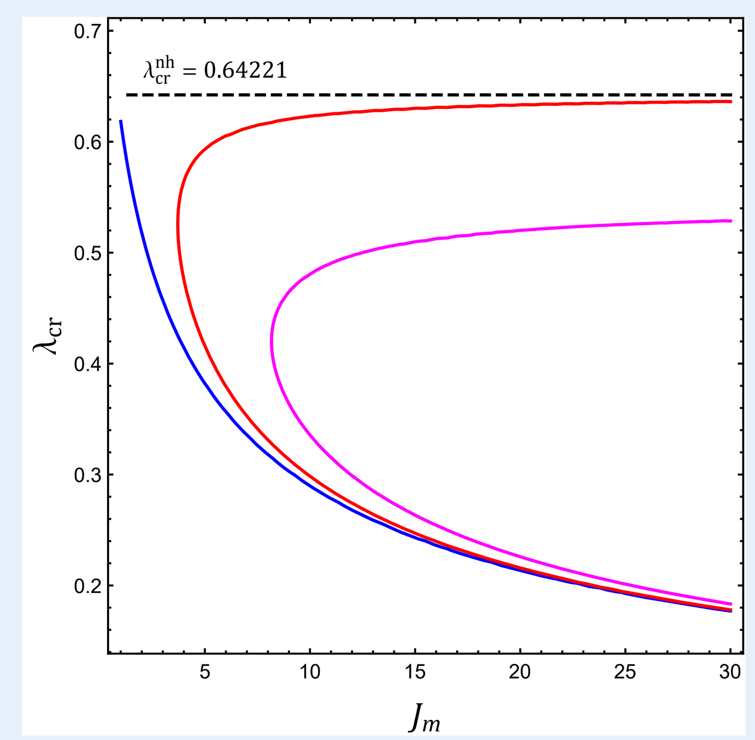

Figure 4: The variation of critical stretches λ_{cr} against J_m for wrinkling (pink), creasing (red) and rigidity (blue). Creases are absent for $J_m < 3.186$. In the limit $J_m \to \infty$, the neo-Hookean model $\bar{W}(I_1) = \frac{1}{2}\mu\left(I_1 - 2\right)$ is recovered, and $\lambda_{cr} = 0.64221$.